

# Matrix models vs. Seiberg–Witten/Whitham theories

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## Abstract

We discuss the relation between matrix models and the Seiberg–Witten type (SW) theories, recently proposed by Dijkgraaf and Vafa. In particular, we prove that the partition function of the Hermitean one-matrix model in the planar (large  $N$ ) limit coincides with the prepotential of the corresponding SW theory. This partition function is the logarithm of a Whitham  $\tau$ -function. The corresponding Whitham hierarchy is explicitly constructed. The double-point problem is solved.

1. It is well known that partition functions of matrix models are  $\tau$ -functions of integrable hierarchies of the Toda type [1]. In the specific double scaling limit, these  $\tau$ -functions become  $\tau$ -functions of various reduction of the KP hierarchy [2]. If one makes the simplest, large- $N$  (planar) limit, the partition function becomes the  $\tau$ -function of the dispersionless Toda hierarchy, which in turn becomes the  $\tau$ -function of the dispersionless (reductions of) KP hierarchy [3] after performing the continuum limit (which basically means working nearby a singularity of the partition function). All these dispersionless hierarchies are just Whitham equations over trivial solutions to integrable (Toda, KP) hierarchies.

When solving matrix models, most attention was paid to one-cut solutions where the limiting eigenvalue distribution spans one interval on the real axis [4]. The results on multi-cut solutions were few [5, 6, 7]. Recently, Dijkgraaf and Vafa proposed [8] the new insight on the multi-cut large- $N$  limit of matrix models. Namely, they associated this limit with a Riemann surface and some related SW system. Its prepotential, which we prove here to be the logarithm of the large- $N$  partition function, is typically associated with the logarithm of some Whitham  $\tau$ -functions [9, 10]. This hints that the matrix model in the large  $N$  limit of multi-cut type describes the Whitham system over a non-trivial, finite-gap solution to integrable (Toda, KP) hierarchy. In particular, this solution passes to a finite-gap solution of (reductions of) the KP hierarchy in the continuum limit.

In this paper, we restrict ourselves with the simplest example of the Hermitean one-matrix model. We show that coefficients of the potential of the model gives rise to Whitham flows and manifestly construct this Whitham system. In fact, the authors of [8] associated the  $N = 1$  SUSY gauge theory studied in [11] with the SW system related to the multi-cut planar limit of matrix models. From the point of view of  $N = 1$  SUSY theory, these coefficients must be identified with couplings in the tree superpotential, while the SW moduli are associated with v.e.v.'s of the gluino condensates. This gives an interpretation of the results of [11] in the Whitham hierarchy terms.

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2. We call SW system [12]<sup>1</sup> the following set of data<sup>2</sup>:

- a family  $\mathcal{M}$  of Riemann surfaces (complex curves)  $\mathcal{C}$  whose dimension coincides with the genus;<sup>3</sup>
- a meromorphic differential  $dS$  whose variations w.r.t. moduli of curves are holomorphic.

This data allows one to define the notion of prepotential [12, 16] related to some integrable system [9].

Indeed, one can introduce variables

$$a_i \equiv \oint_{A_i} dS \quad (1)$$

where  $A_i$  are  $A$ -cycles on  $\mathcal{C}$ . Then,

$$d\omega_i \equiv \frac{\partial dS}{\partial a_i} \quad (2)$$

are canonical holomorphic differentials on  $\mathcal{C}$  (normalized so that  $\oint_{A_i} d\omega_j = \delta_{ij}$ ). Then, introducing  $B$ -cycles conjugated to  $A$ -cycles:  $A_i \circ B_j = \delta_{ij}$ , where  $\circ$  means intersection, we obtain that

$$\frac{\partial}{\partial a_i} \oint_{B_j} dS = \oint_{B_i} d\omega_j = T_{ij} \quad (3)$$

is the period matrix of  $\mathcal{C}$  and is therefore symmetric. Hence, there exists a prepotential  $\mathcal{F}$  such that

$$\frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} dS \quad (4)$$

3. Let us consider the Hermitean one-matrix model. Its partition function is given by the integral over Hermitean  $N \times N$  matrix  $M$

$$Z_N = \int \mathcal{D}M e^{-N \text{tr} V(M)} \quad (5)$$

where  $\mathcal{D}M$  is the Haar measure on Hermitean matrices and the potential  $V(x)$  is a polynomial of degree  $n+1$ . After integrating out angular variables, we obtain [17]

$$Z_N \sim \int \prod_i d\lambda_i e^{-NV(\lambda_i) + \sum_{j \neq i} \log(\lambda_i - \lambda_j)} \quad (6)$$

In the large  $N$  limit, it is standard to introduce the density of eigenvalues

$$\rho(\lambda) \equiv \frac{1}{N} \sum_i \delta(\lambda - \lambda_i). \quad (7)$$

Then, (6) can be rewritten as

$$Z_N \sim \int \prod_i d\lambda_i e^{-N^2 \left[ \int \rho(\lambda) V(\lambda) d\lambda - \iint \rho(\lambda) \rho(\lambda') \log(\lambda - \lambda') d\lambda d\lambda' \right]} \quad (8)$$

In the large  $N$  limit, this integral can be evaluated by the saddle point method. We then assume  $\rho(\lambda)$  to be a continuous function such that

$$\rho(\lambda) \geq 0, \quad \int \rho(\lambda) d\lambda = 1 \quad (9)$$

We then obtain the saddle point equation

$$V(\lambda) + \xi = 2 \int \rho(\lambda') \log(\lambda - \lambda') d\lambda', \quad \lambda, \lambda' \in \text{supp}(\rho) \quad (10)$$

where the support of the function  $\rho(\lambda)$  comprises  $\lambda$  such that  $\rho(\lambda) \neq 0$ . It emerges in this equation because of the first condition in (9). The constant  $\xi$  in equation (10) is just the Lagrange multiplier for the second condition in (9).

<sup>1</sup>Various properties of such systems can be found in [13, 14].

<sup>2</sup>Our definition of the SW prepotential does not imply any connection with prepotentials of  $N=2$  SUSY gauge theories [14]. Moreover, the prepotentials discussed in this paper are rather related to superpotentials of  $N=1$  SUSY theory [11].

<sup>3</sup>This restriction can be waved, see examples in [15].

However, in order to use analytic tools (the Cauchy problem), we must investigate not eq.(10) but its derivative

$$V'(\lambda) = 2 \int \frac{\rho(\lambda')}{\lambda - \lambda'} d\lambda', \text{ or } V'(\lambda) = \oint \frac{\rho(\lambda')}{\lambda - \lambda'} d\lambda'. \quad (11)$$

In order to solve this equation, we introduce the function

$$y(\lambda) \equiv \frac{2}{N} \sum_i \frac{1}{\lambda_i - \lambda} + V'(\lambda) = 2 \int \frac{\rho(\lambda')}{\lambda' - \lambda} d\lambda' + V'(\lambda) \quad (12)$$

such that its imaginary part coincides with  $\rho(\lambda)$  because of (11). In the large  $N$  limit, it satisfies the equation

$$y^2(\lambda) - V'^2(\lambda) + 4 \int \frac{V'(\lambda) - V'(\lambda')}{\lambda - \lambda'} \rho(\lambda') d\lambda' \equiv y^2(\lambda) - V'^2(\lambda) + f_{n-1}(\lambda) = 0 \quad (13)$$

where  $f_{n-1}(\lambda)$  is a polynomial of degree  $n - 1$ . This means that the general solution to (11) is

$$y^2 = V'^2(\lambda) - f_{n-1}(\lambda) = \prod_{i=1}^{2n} (\lambda - \mu_i) \quad (14)$$

This equation describes a hyperelliptic curve of genus  $n - 1$ . It is, however, not arbitrary for a fixed potential  $V(\lambda)$ , because it follows from (13) that

$$y(\lambda) - V'(\lambda) = W(\lambda) \sim \frac{2}{\lambda} + O(\lambda^{-2}) \Big|_{\lambda \rightarrow \infty}, \quad (15)$$

i.e.,  $\mu_i$  are not independent. Here  $W(\lambda)$  is the standard loop mean [4]. One more restriction comes from the normalization condition in (9), and one is left with  $n - 1$  free moduli. This is exactly what we need for SW system given on curve (14).

A solution to eq.(11) is parameterized by  $n - 1$  moduli, i.e., these moduli span the moduli space of planar limits of the matrix model. The function  $\rho(\lambda)$ , which is imaginary part of  $y(\lambda)/(2\pi)$ , has the support on  $n$  different branching cuts.

Note that within the standard matrix model framework, there are two more requirements that leave no moduli in solution. First of all, one can easily see that the sign of  $\rho(\lambda)$  changes when coming to the next cut. This spoils non-negativity of  $\rho(\lambda)$  and means that solution (14) is not stable. This means that the cuts with  $\rho(\lambda)$  negative must shrink to produce double points. Thus,  $y(\lambda)$  becomes proportional, besides the square root of a polynomial, to some other polynomial that has odd numbers of zeroes between cuts and, therefore, changes sign on every next cut.

We shall explain below that one can easily include these double points into the general SW and Whitham framework. Moreover, they allow one to construct more general Whitham systems.<sup>4</sup>

The second requirement looks more fundamental. Namely, returning to original equation (10), one has to check that the Lagrange multiplier  $\xi$  is the same for *every* cut (while (11) only guarantees it is a constant on a cut). The difference of values of  $\xi$  on two neighbour cuts is equal to [5]

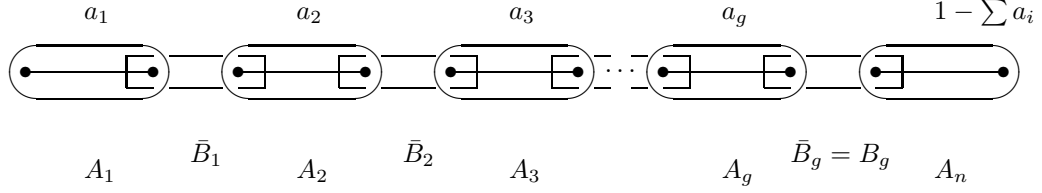
$$\xi_{i+1} - \xi_i = \int_{\mu_{2i}}^{\mu_{2i+1}} y(\lambda) d\lambda \quad (16)$$

where the integral runs from the right end of the left cut to the left end of the right cut. This gives  $n - 1$  additional constraints and leaves no moduli (there still remains a freedom in the number of cuts). We must wave this requirement in order to make the Whitham system nontrivial. So, instead of just matrix models, it is better to speak about matrix-model-like solutions of Cauchy problem (11). So, at the moment we just ignore this last restriction and work with solutions to eq.(11), which we call matrix model solutions. We shall return to this point later.

**4.** Now we associate an SW system with the planar limit of the matrix model. The family of genus  $g$  curves is described by eq.(14) with the restriction (15). We describe it on the complex plane  $\lambda$  by  $n = g + 1$  cuts, Fig.1. Besides canonically conjugated  $A$ - and  $B$ -cycles, we also use the linear combination of  $B$ -cycles:  $\bar{B}_i \equiv B_i - B_{i+1}$ ,  $\bar{B}_{n-1} \equiv B_{n-1}$ . Therefore,  $\bar{B}$ -cycles encircle the nearest ends of two neighbour cuts, while all  $B$ -cycles goes from

<sup>4</sup>V.Kazakov suggested to overcome non-stability of solutions without double points via some proper analytic continuation.

a given right end of the cut to the last,  $n$ -th cut. For the sake of definiteness, we order all points  $\mu_i$  in accordance with their index so that  $\mu_i$  is to the right of  $\mu_j$  if  $i > j$ .



**Fig. 1.** Structure of cuts and contours.

The SW differential is

$$dS = y(\lambda)d\lambda \quad (17)$$

Its variations w.r.t. moduli is holomorphic on  $\mathcal{C}$  (13) because all moduli are hidden in the polynomial  $f_{n-1}(\lambda)$ :

$$\frac{\partial dS}{\partial \text{moduli}} = \frac{\partial f_{n-1}(\lambda)}{\partial \text{moduli}} \frac{d\lambda}{y} \quad (18)$$

This expression is holomorphic, because the leading coefficient of  $f_{n-1}(\lambda)$  is fixed by the normalization condition (9), and the differentials  $\lambda^k \frac{d\lambda}{y}$  are holomorphic on the curve  $\mathcal{C}$  (13) for  $k = 0, 1, \dots, n-2$ . Therefore, we introduce the variables

$$a_i = \frac{1}{2} \oint_{A_i} y(\lambda)d\lambda = \text{Im} \int_{\mu_{2i-1}}^{\mu_{2i}} y(\lambda)d\lambda = \int_{\mu_{2i-1}}^{\mu_{2i}} \rho(\lambda)d\lambda, \quad i = 1, \dots, n-1 \quad (19)$$

that have meaning of “the occupation numbers” (or numbers of eigenvalues) associated with a given cut. Note also that

$$\frac{1}{2} \oint_{A_n} y(\lambda)d\lambda = \text{Im} \int_{\mu_{2n-1}}^{\mu_{2n}} y(\lambda)d\lambda = 1 - \sum_{i=1}^{n-1} a_i \quad (20)$$

which follows from (9). This is exactly the condition that fixes the leading coefficient of  $f_{n-1}(\lambda)$  and leaves  $n-1$  moduli. It means that

$$\frac{1}{2} \oint_{A_n} d\omega_i = \frac{1}{2} \oint_{A_n} \frac{\partial dS}{\partial a_i} = -1 \quad \text{for all } i \quad (21)$$

Now one defines the prepotential

$$\frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} dS \quad (22)$$

This prepotential is equal to logarithm of the matrix model partition function,  $\log Z_N$  in the planar limit. Indeed, let  $\mathcal{D}$  be a set of contours  $A_1 \cup A_2 \cup \dots \cup A_g \cup A_n$ . The large  $N$  partition function is<sup>5</sup>

$$\log Z_N = -\frac{1}{2} \oint_{\mathcal{D}} y(\lambda)V(\lambda)d\lambda + \frac{1}{4} \oint_{\mathcal{D}} \oint_{\mathcal{D} \times \mathcal{D}} y(\lambda) \log(\lambda - \lambda') y(\lambda') d\lambda d\lambda' \quad (23)$$

We now calculate the derivative of  $\log Z_N$  w.r.t.  $a_i$ :

$$\frac{\partial \log Z}{\partial a_i} = -\frac{1}{2} \int_{\mathcal{D}} d\lambda \frac{\partial y(\lambda)}{\partial a_i} \left( V(\lambda) - \int_{\mathcal{D}} d\lambda' \log(\lambda - \lambda') y(\lambda') \right) \quad (24)$$

The expression in the brackets on the rhs of (24) is a step function, which is equal to  $\xi_i$  on each cut  $A_i$  and its values on different cuts are (16)

$$\xi(\lambda) \equiv V(\lambda) - \int_{\mathcal{D}} d\lambda' \log(\lambda - \lambda') y(\lambda') = \begin{cases} \xi_1 \equiv h_1 \text{ for } \lambda \in A_1, \\ \xi_1 + \oint_{\bar{B}_1} y(\lambda') d\lambda' \equiv h_2 \text{ for } \lambda \in A_2, \\ \vdots \\ \xi_1 + \oint_{\bar{B}_1 \cup \bar{B}_2 \cup \dots \cup \bar{B}_{g-1}} y(\lambda') d\lambda' \equiv h_g \text{ for } \lambda \in A_g, \\ \xi_1 + \oint_{\bar{B}_1 \cup \bar{B}_2 \cup \dots \cup \bar{B}_{g-1} \cup \bar{B}_g} y(\lambda') d\lambda' \equiv h_n \text{ for } \lambda \in A_n. \end{cases} \quad (25)$$

<sup>5</sup>From now on, we consider symbols  $\oint$  and  $\text{res}$  with additional factors  $(2\pi i)^{-1}$  so that  $\text{res}_0 \frac{d\xi}{\xi} = -\text{res}_\infty \frac{d\xi}{\xi} = \oint \frac{d\xi}{\xi} = 1$ .

We therefore have

$$\frac{\partial \log Z_N}{\partial a_i} = - \int_{\mathcal{D}} \frac{\partial dS}{\partial a_i} h(\lambda) = - \int_{\mathcal{D}} \omega_i h(\lambda) = -\xi_i + \xi_n = \oint_{\bar{B}_i \cup \bar{B}_{i+1} \cup \dots \cup \bar{B}_g} dS \equiv \oint_{B_i} dS \quad (26)$$

and  $Z_N$  in the planar limit can be, indeed, identified with the prepotential  $e^{\mathcal{F}}$ .

**5.** One can learn two lessons from this fact. First of all, we can return to the interpretation of different  $\xi_i$  on different cuts within matrix model. The standard matrix model case of equal  $\xi_i$ 's can be now formulated as the set of conditions

$$\frac{\partial \mathcal{F}}{\partial a_1} = \dots = \frac{\partial \mathcal{F}}{\partial a_g} = 0 \quad (27)$$

These are the conditions of minimum of the matrix model partition function w.r.t. the occupation numbers. They can be removed by introducing different chemical potentials for different cuts.<sup>6</sup> We do not enter here any further details and go instead to another lesson.

We know from studies of matrix models that their partition functions are  $\tau$ -functions of some integrable hierarchies [1]. What are they in the planar limit? We have just proved that such a partition function is an SW prepotential in this limit. One typically associates logarithms of Whitham  $\tau$ -functions with SW prepotentials [9, 10]. Therefore, we may expect that the matrix model partition function becomes the  $\tau$ -function of some Whitham hierarchy. An additional evidence for this comes from looking at the simplest one-cut large- $N$  solution of the matrix model, when the partition function becomes the  $\tau$ -function of the dispersionless Whitham hierarchy [3]. Now we construct this hierarchy in very manifest terms.

First, we return to the problem of double points. Let us assume that some of the cuts shrink, i.e.,

$$y(\lambda) = M_{n-k}(\lambda) \sqrt{\prod_{i=1}^{2k} (\lambda - \mu_i)} \equiv M_{n-k}(\lambda) \sqrt{g_{2k}(\lambda)} \quad (28)$$

where  $M_{n-k}(\lambda)$  is a polynomial of degree  $n - k$  and  $g_{2k}(\lambda)$  is a polynomial of degree  $2k$ . This means that one is effectively left with a new curve

$$y(\lambda) = \sqrt{g_{2k}(\lambda)} \quad (29)$$

This curve of lower genus  $k - 1$  along with the differential  $dS = M_{n-k}(\lambda) y(\lambda) d\lambda$  remarkably give rise to a new SW system that depends on  $k - 1$  moduli.

To see this, one needs to take into account that there still holds eq.(13),

$$y^2(\lambda) M_{n-k}^2(\lambda) = V'^2(\lambda) - f_{n-1}(\lambda) \equiv V'^2(\lambda) - 2(V'(\lambda)W(\lambda))_+, \quad (30)$$

where we let  $(\cdot)_+$  denote the polynomial part of the expression in brackets. Then, varying  $dS$  and using (29), we obtain for the *general* variation  $\delta dS$ :

$$\delta dS = \delta (M_{n-k}(\lambda) y(\lambda)) d\lambda = \frac{g_{2k}(\lambda) \delta M_{n-k}(\lambda) + \frac{1}{2} M_{n-k}(\lambda) \delta g_{2k}(\lambda)}{y} d\lambda \quad (31)$$

On the other hand, doing a variation  $\tilde{\delta}$  of  $M_{n-k}(\lambda) y(\lambda)$  that does not alter the potential, we obtain from (30) that

$$\tilde{\delta} dS = -\frac{1}{2} \frac{\tilde{\delta} f_{n-1}(\lambda)}{M_{n-k}(\lambda) y(\lambda)} d\lambda. \quad (32)$$

Because this variation is a particular case of (31), we obtain that zeroes of  $M_{n-k}(\lambda)$  in the denominator of (32) must *cancel*, so the maximum degree of the polynomial in the numerator is  $n - 2$ . The variation is then *holomorphic* on curve (29).

This solves the problem of double points. The corresponding system with double points (the large- $N$  limit of the matrix model) is still described by the SW theory.

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<sup>6</sup>Putting differently, one can interpret these conditions as a criterium of stability against tunneling of eigenvalues between different cuts [18]. Stability is achieved by imposing equality of the chemical potentials of all cuts.

**6.** Let us return to the case  $n = k$ . This SW system is described by  $n - 1$  and  $n + 1$  additional parameters. They are expected to be Whitham times giving flows on the moduli of the finite gap solution. In this moduli space there are deformations leaving the curve within the family (variations of  $f_{n-1}(\lambda)$  that does not change the potential), and those transversal to the family. These latter are defined by the potential. If one stays within the family of hyperelliptic curves (13), there are exactly  $n + 1$  transversal deformations. Therefore, in order to have enough many deformations (Whitham times), one needs to involve potentials of high enough degree, i.e. to deal with the construction with double points.

In our manifest construction of the Whitham system we mainly follow [10, 19] (see also [20]). In order to construct a Whitham system, one needs to add to the SW data a set of punctures with local coordinates in their vicinity. These points here are the two infinities on the curve (29) and the local parameter is  $\eta = \frac{1}{\lambda}$ . Now one introduces a set of meromorphic differentials  $d\Omega_n$  with the poles only at punctures (since the hyperelliptic curve (29) is invariant w.r.t. the involution  $y \rightarrow -y$ , from now on we just work with either of the two infinities, see [10, 19]) and the behaviour

$$d\Omega_m = (\eta^{-m-1} + O(1)) d\eta, \quad \eta \rightarrow 0 \quad (33)$$

Then, the Whitham system is generated by a set of equations for these differentials and the holomorphic differentials  $d\omega_i$ :

$$\frac{\partial d\Omega_p}{\partial t_m} = \frac{\partial d\Omega_m}{\partial t_p}, \quad \frac{\partial d\Omega_m}{\partial a_i} = \frac{\partial d\omega_i}{\partial t_m}, \quad \frac{\partial d\omega_i}{\partial a_j} = \frac{\partial d\omega_j}{\partial a_i} \quad (34)$$

These equations implies that there exists a differential  $dS$  such that

$$\frac{\partial dS}{\partial a_i} = d\omega_i, \quad \frac{\partial dS}{\partial t_m} = d\Omega_m \quad (35)$$

Let us check that the differential  $dS \equiv M_{n-k}(\lambda)y(\lambda)d\lambda$  given on the curve (29) with the relation for moduli (30) really satisfies (35).

Indeed, we have proved the first set of relations (35) in the previous paragraph. Now let us consider variations of the potential, i.e., variations w.r.t. Whitham times  $t_m$ . Then, we obtain instead of (32)

$$\delta dS = -\frac{1}{2} \frac{\delta (V'^2(\lambda) - f_{n-1}(\lambda))}{M_{n-k}(\lambda)y(\lambda)} d\lambda \quad (36)$$

while (31) still holds. Repeating the argument of the previous paragraph, we conclude that the zeroes of  $M_{n-k}(\lambda)$  cancel from the denominator and, therefore, the variation may have pole only at  $\lambda = \infty$  or  $\eta = 0$ , i.e. at the puncture. In order to estimate this pole, one needs to use (31), which implies that  $dS = M_{n-k}(\lambda)y(\lambda)d\lambda \rightarrow (V'(\lambda) + O(\frac{1}{\lambda}))d\lambda$  and, therefore, the variation of  $dS$  at large  $\lambda$  is completely determined by the variation of  $V'(\lambda)$ . Parameterizing  $V(\lambda) = \sum_{m=1}^{n+1} t_m \frac{\lambda^m}{m}$  one comes to (35) up to a linear combination of holomorphic differentials. One may fix the normalization of  $d\Omega_m$  that are also defined up to a linear combination of holomorphic differentials so that eq.(35) would be *exact*. This is achieved merely by imposing the (obvious) condition

$$\frac{\partial a_i}{\partial t_m} = 0 \quad \forall i, m, \quad (37)$$

where  $a_i$  are defined by (19).

Thus, similarly to eq.(1) we can invariantly introduce variables  $t_m$  via the relation

$$t_m = \text{res}_{\eta=0} \eta^m dS \quad (38)$$

and define the prepotential that depends on both  $a_i$  and  $t_m$  via the old relation (4) and the similar relation

$$\frac{\partial \mathcal{F}}{\partial t_m} = \frac{1}{m} \text{res}_{\eta=0} \eta^{-m} dS \quad (39)$$

One can immediately prove that such a prepotential exists [10, 19], i.e., the second derivatives are symmetric, and, moreover, similarly to Sec. 4, we find that thus defined  $\mathcal{F}$  coincides with  $\log Z_N$  in the planar limit. For this, we apply the formula similar to (24) with the only difference that the potential  $V(\lambda)$  itself is changed. We then obtain from (25)

$$\frac{\partial \log Z}{\partial t_m} = -\frac{1}{2} \int_{\mathcal{D}} d\lambda \frac{\partial y(\lambda)}{\partial t_m} \cdot \xi(\lambda) - \frac{1}{2} \int_{\mathcal{D}} d\lambda y(\lambda) \frac{\lambda^m}{m} = -\sum_{i=1}^g \frac{\partial a_i}{\partial t_m} (h_i - h_n) + \frac{1}{m} \text{res}_{\eta=0} \eta^{-m} dS, \quad (40)$$

which by virtue of (37) gives (39). Therefore,  $Z_N$  in the planar limit is the Whitham  $\tau$ -function, and the whole machinery of Whitham systems works here in full strength.

7. After having constructed the large  $N$  (planar) limit, next step is to take the (double scaling) continuum limit. Namely, one has to dwell nearby a singularity (branching point) of  $\rho(\lambda)$ . Say, one can work nearby the left end of the very right,  $n$ -th cut  $\mu_{2n-1}$  [4]. The standard argument then is that one feels no other cuts, since they are far away. This would mean that multi-cut solutions coincide with the one cut solution in the continuum limit. This is, however, the case only if all the other branching points do not come close to  $\mu_{2n-1}$ . Otherwise, there exist non-trivial continuum limits [7]. In the most non-trivial situation, all the branching points but  $\mu_{2n}$  come close to each other<sup>7</sup>. This is equivalent just to sending  $\lambda_{2n}$  to infinity. Such a curve describes a finite gap solution to the KdV hierarchy, moreover, the corresponding SW system is also associated with KdV [21]. Therefore, we expect that the matrix model partition function in this limit describes (in leading order) the Whitham hierarchy over KdV finite gap solution.

However, it would be very instructive to construct an entire double scaling limit in this situation, in particular, to fix proper scaling behaviours. This means to match properly the growth of  $N$  and approaching to the singularity. Then, one could address the problem of exact (matrix model)  $\leftrightarrow$  (SW) correspondence, in particular, in integrable terms. In particular, it would be interesting to see what is the proper deformation of SW systems in this case.

In this respect, the problem of studying higher-genus corrections looks very natural because the whole matrix-model-like solution (the solution to the loop equation, see [4]) in all genera is completely determined by the set of data  $\{a_i, t_m\}$  and because it was proved [22] that spectral correlators manifest the universality property for multi-cut solutions as well. The integrable system that will appear in this approach must be a generalization of a Whitham system.

At last, let us note that the construction considered in this paper is directly extendable to other matrix models. In particular, one can consider the model of normal matrix that has much to do with the problem of Laplacian growth [23]. In fact, its naive large  $N$  limit describes how external and internal moments of a domain are related. This domain is a counterpart of the one cut. Moreover, the system is described by the Whitham hierarchy that is the dispersionless Toda system. Now, considering a multi-domain solution and introducing chemical potentials for different domains, one has to get the Whitham system over a finite-gap solution to the Toda system. This Whitham hierarchy should relate the external and internal moments of several domains.

The authors are grateful to A.Gorsky, S.Gukov, V.Kazakov, A.Marshakov and especially to A.Morozov for fruitful discussions.

We acknowledge the remarkable atmosphere at Mologa meeting where this work has been started.

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<sup>7</sup>Note that the authors of the paper [8] considered the branching point  $\mu_{2n-1}$  also located far away, at some large distance  $\Lambda$ . Then, they were interested in the pieces of the prepotential that shows up a logarithmic behaviour at  $\Lambda \rightarrow \infty$  plus terms constant in  $\Lambda$ . This degenerated situation better suits the  $N = 1$  SUSY theory needs, where  $\Lambda$  plays a role of the  $\Lambda_{QCD}$  parameter [11]. In order to get this limit, one just suffices to rescale  $\rho(\lambda) \rightarrow \frac{\rho(\lambda)}{\sqrt{\lambda-\Lambda}}$  with all the  $B$ -cycles still encircling the point  $\Lambda$ . Thus, the number of cuts becomes  $n - 1$ , but still there is a puncture at infinity at  $\Lambda$ . With properly subtracted  $\Lambda$ -divergent pieces, it gives the same function  $\mathcal{F}(a)$ , with  $a_i$  just rescaled. This gives the calculational recipe of the paper [11]. The only subtlety is that, in order to properly cut-off logarithmic terms, more concretely, to reproduce the terms  $a_i^2 \log \Lambda$  in the prepotential, one needs to rescale properly the matrix model partition function by volumes of the unitary groups, see [8].

For instance, the simplest non-trivial example of the two-cut solution

$$y = \sqrt{(\lambda^2 - \mu^2)(\lambda - \Lambda)} \quad (41)$$

reduces in this limit to the semi-circle distribution  $y = \sqrt{\lambda^2 - \mu^2}$ , but the prepotential is determined by the integral  $\frac{\partial \mathcal{F}}{\partial a} = \int_{\mu}^{\Lambda} \sqrt{\lambda^2 - \mu^2} d\lambda$  with  $a = \int_{-\mu}^{\mu} \sqrt{\lambda^2 - \mu^2} d\lambda$ . This immediately gives  $\mathcal{F} \sim a^2 \log \frac{a}{\Lambda^2} + \dots$  and coincides with the result obtained by exact calculating with (41) and then bringing  $\Lambda$  to infinity, although  $a$  gets rescaled by  $\sqrt{\Lambda}$ .

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